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On non-radially symmetric solutions of the Liouville-Gel'fand equation on a two-dimensional annular domain

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1 Introduction

We consider the Liouville-Gel'fand equation

$$\begin{cases} \Delta u+\lambda e^{u}=0 & \text{in} \quad \Omega_{\varepsilon}, \\ u=0 & \text{on} \quad \partial\Omega_{\varepsilon}, \end{cases} \tag{LG}$$

where λ is a positive parameter and Ω_{ε} is a two-dimensional annulus defined by

$$\Omega_{\varepsilon}:=\left\{x\in \mathbb{R}^{2};\ \varepsilon<|x|<1\right\}$$

for $0<\varepsilon<1$. What we are concerned with is the structure of non-radially symmetric solutions of (LG) when ε is small.

If a domain is a disk, from the well known result obtained by Gidas, Ni and Nirenberg [5], there is no non-radially symmetric solution of (LG). On the other hand in the case of an annulus, the existence of non-radially symmetric solutions is revealed by Lin [7] and Nagasaki and Suzuki [8]. More precisely, Lin showed that non-radially symmetric solutions appear through a bifurcation from radially symmetric solutions and Nagasaki and Suzuki proved that for any $k\in \mathbb{N}$, there exists a k-mode solution such that $\int_{\Omega}e^{u}dx$ is large. Here, by k-mode solution, we mean a solution which is invariant under the rotation of $2\pi/k$, and is not invariant under the rotation of $2\pi/m$ for $m>k$. From the subsequent work by Dancer [2], the set of the bifurcating non-radially symmetric solutions is unbounded in (λ, u) plane. Additionally, for a general non-simply connected domain, del Pino, Kowalczyk and Musso [3] obtained a solution which blows up at k different points as $\lambda\to 0$.

From these results, it is expected that the bifurcating non-radially symmmetric solutions connect to the large solutions obtained in [8, 3]. Our problem is to show this expectation when the inside radius of the annulus is small.

To accomplish this, first we have to derive an appropriate limiting equation of (LG) as $\varepsilon\to 0$ and study (non-radially symmetric) solutions of the limiting equation. These were investigated in [6]. We will introduce the limiting equation and mention the relation between (LG) and the limiting equation in the next section briefly. Based on the study of the limiting equation, we construct solutions of (LG).

2 Limiting equation and Main result

In this section we introduce the limiting equation of (LG) obtained in [6] and state our main result. The limiting equation is given by the following.

$$\begin{cases} \Delta v + Ae^v = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}, \\ v(x) = \begin{cases} (B-2)\log|x| + o(1) & \text{as} \quad |x| \to 0, \\ -(B+2)\log|x| + o(1) & \text{as} \quad |x| \to \infty, \end{cases} \end{cases} \tag{LE}$$

where $A > 0$ and $B \geq 2$ are parameters. This equation is derived by the method of matched asymptotic expansions. See [6] for details. We only explain that an approximate solution of (LG) can be constructed if we find a solution of (LE). Let v be a solution of (LE) and put

$$\Lambda := A\varepsilon^{\left(\frac{B}{2}-1\right)},$$
$$U(x) := \left(\frac{B}{2} - \frac{2}{B}\right)\log\frac{1}{\varepsilon} + v\left(\varepsilon^{-\left(\frac{1}{2}-\frac{1}{B}\right)}x\right).$$

Then we see at once that (Λ, U) satisfies

$$\Delta U + \Lambda e^U = 0 \quad \text{in} \quad \Omega_\varepsilon.$$

Furthermore, the bottom equation of (LE) implies that as $\varepsilon \to 0$,

$$U(x) = \begin{cases} \left(\frac{B}{2} - \frac{2}{B}\right)\log\frac{1}{\varepsilon} + (B-2)\log\varepsilon^{\frac{1}{2}+\frac{1}{B}} + o(1) = o(1) & \text{if} \quad |x| = \varepsilon, \\ \left(\frac{B}{2} - \frac{2}{B}\right)\log\frac{1}{\varepsilon} - (B+2)\log\varepsilon^{-\left(\frac{1}{2}-\frac{1}{B}\right)} + o(1) = o(1) & \text{if} \quad |x| = 1 \end{cases}$$

provided that $B > 2$. This says that U approximately satisfies the boundary condition of (LG). Therefore $(\lambda, u) = (\Lambda, U)$ is an approximate solution of (LG).

We introduce solutions of (LE). Radially symmetric solutions of (LE) are given by

$$(A, B, v) = (8K^2, 2K, v_K), \quad v_K(r) = \log\frac{1}{r^2(r^K + r^{-K})^2},$$

where $r = |x|$ and $K \geq 1$ is a parameter. Moreover, (LE) has the following non-radially symmetric solutions.

$$(A, B, v) = (8k^2(1-\rho^2), 2k, v_{k,\rho,\gamma}), \quad v_{k,a,\gamma}(x) = \log\frac{1}{r^2\{r^k + r^{-k} - 2\rho\cos(k\theta + \gamma)\}^2}.$$

Here $x = (r\cos\theta, r\sin\theta)$, $k \in \mathbb{N}$, $\rho \in (0,1)$ and $\gamma \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Parameters k, ρ and γ represent the number of frequency in the rotational direction, dilation and rotation respectively. This non-radially symmetric solution was first exhibited in [1], and it was shown in [9] that all the solutions of (LE) consist only of the above radially and non-radially symmetric solutions. See also [6].

The approximate solution (Λ, U) of (LG) by using the above non-radially symmetric solution is $(\Lambda, U) = (8k^2(1-\rho^2)\varepsilon^{k-1}, (k-1/k)\log(1/\varepsilon) + v(\varepsilon^{-\frac{k-1}{2k}}x))$. This function approximately satisfies (LG) provided that $k \geq 2$, while this approximation fails if $k = 1$. Therefore we have to modify the approximation in this case, and this actually can be done. The following theorem is our main result, which concerns the construction of solutions of (LG) based on the approximate solutions.

Theorem 1. *Let $\delta > 0$ be an arbitrary fixed constant. Then there exists a positive number ε_0 such that, for any $\varepsilon \in (0, \varepsilon_0]$,* (LG) *has non-radially symmetric solutions*

$$(\lambda, u) = (8k^2(1-\rho^2)\varepsilon^{k-1}, u_{\varepsilon,k,\rho,\gamma}), \quad k \in \mathbb{N}, \rho \in [\delta, 1-\delta], \gamma \in S^1$$

which satisfies

$$u_{\varepsilon,k,\rho,\gamma}(x) = \begin{cases} \left(k - \dfrac{1}{k}\right)\log\dfrac{1}{\varepsilon} + v_{k,\rho,\gamma}\left(\varepsilon^{-\frac{k-1}{2k}}x\right) + O\left(\varepsilon^{\frac{k-1}{2}}\right) & \text{if } k \geq 2, \\ 4\log\dfrac{1}{\tau_\varepsilon} + v_{1,\rho,\gamma}\left(\tau_\varepsilon^{-1}x\right) + O\left(\tau_\varepsilon \log\dfrac{1}{\tau_\varepsilon}\right) & \text{if } k = 1 \end{cases}$$

as $\varepsilon \to 0$. Here $\tau_\varepsilon > 0$ is the solution of the equation $(2\log\tau)/\tau = \log\varepsilon$, and the above expansion is uniform for $x \in \Omega_\varepsilon$, $k \in \mathbb{N}$, $\rho \in [\delta, 1-\delta]$ and $\gamma \in S^1$.

This theorem indicates that non-radially symmetric solutions bifurcating from radially symmetric solutions connect to the large solutions obtained in [8, 3], as we expected.

In the next section, we discuss how Theorem 1 is proved.

3 Sketch of proof

We mention the sketch of the proof of Theorem 1 in this section. We only treat the case $k \geq 2$. By setting $\lambda = 8(1-\rho^2)\varepsilon^{k-1}$ and performing the change of variables $x \mapsto \varepsilon^{\frac{k-1}{2k}}x$, (LG) is rewritten as

$$\begin{cases} \Delta u + 8k^2(1-\rho^2)\varepsilon^{k-1/k}e^u = 0 & \text{in } \tilde{\Omega}_\varepsilon, \\ u = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon, \end{cases} \tag{3.1}$$

where

$$\tilde{\Omega}_\varepsilon := \left\{x \in \mathbb{R}^2; \varepsilon^{\frac{1}{2}\left(1+\frac{1}{k}\right)} < |x| < \varepsilon^{-\frac{1}{2}\left(1-\frac{1}{k}\right)}\right\}.$$

We introduce a correction function to correct the boundary value of the approximate solution. The correction function v_c is defined as a solution of the linear equation

$$\begin{cases} \Delta v_c = 0, & \text{in } \tilde{\Omega}_\varepsilon, \\ v_c = -\left(k - \dfrac{1}{k}\right)\log\dfrac{1}{\varepsilon} - v_{k,a,\gamma}, & \text{on } \partial\tilde{\Omega}_\varepsilon. \end{cases}$$

Then one can show that the inequality

$$|v_c(x)| \le C(r^k \varepsilon^{k-1} + r^{-k}\varepsilon^{k+1}) \tag{3.2}$$

holds for some universal constant $C > 0$. Now we substitute $u = \left(k - \frac{1}{k}\right)\log\frac{1}{\varepsilon} + v_{k,a,\gamma} + v_c + v$ and rewrite (3.1) to the equation for v. Then we have

$$\mathcal{L}_{\varepsilon,k,\rho,\gamma}(v) + F_{\varepsilon,k,\rho,\gamma}\ v) = 0, \tag{3.3}$$

where

$$\mathcal{L}_{\varepsilon,k,\rho,\gamma}(v) = \Delta v + 8k^2(1-\rho^2)e^{v_{k,a,\gamma}+v_c}v,$$
$$F_{\varepsilon,k,\rho,\gamma}(v) = 8k^2(1-\rho^2)\left\{e^{v_{k,a,\gamma}+v_c}(e^v - 1 - v) + e^{v_{k,a,\gamma}}(e^{v_c} - 1)\right\}.$$

It is easily seen from (3.2) that

$$|F_{\varepsilon,k,\rho,\gamma}(v)| \le \frac{Ck^2}{r^2(r^k + r^{-k})}\left(|v|^2 + \varepsilon^{k-1}\right) \tag{3.4}$$

provided that $|v| \le 1$. Roughly speaking, the procedure for proving Theorem 1 is that we rewrite (3.3) as $v = -\mathcal{L}^{-1}_{\varepsilon,k,\rho,\gamma}(F_{\varepsilon,k,\rho,\gamma}\ v))$ and then apply the fixed point theorem to this equation in an appropriate function space. Therefore the most important part is the invertibility and the operator norm (in some appropriate space) of $\mathcal{L}_{\varepsilon,k,\rho,\gamma}$. These are ensured by the following lemma.

Lemma 2. *There is a positive constant C depending only on δ such that the inequality*

$$\|\Psi\|_{L^\infty(\tilde{\Omega}_\varepsilon)} \le C\left(\log\frac{1}{\varepsilon}\right)\|\eta_k \mathcal{L}_{\varepsilon,k,\rho,\gamma}(\Psi)\|_{L^\infty(\tilde{\Omega}_\varepsilon)} \tag{3.5}$$

holds for all $k = 2, 3, \ldots$, $\rho \in [\delta, 1-\delta]$, $\gamma \in S^1$ and $\Psi \in \{u \in C^2(\overline{\tilde{\Omega}_\varepsilon}); u = 0$ on $\partial\tilde{\Omega}_\varepsilon\}$ satisfying $\langle \Psi, \Phi_{k,\rho,\gamma,3}\rangle_{L^2(\tilde{\Omega}_\varepsilon, |x|^{-2}dx)} = 0$. Here $\eta_k(x) := \{r^2(r^k + r^{-k})\}/k^2$ and

$$\Phi_{k,\rho,\gamma,3}(x) := \frac{\sin(k\theta+\gamma)}{r^k + r^{-k} - 2\rho\cos(k\theta+\gamma)}$$

for $x \in \mathbb{R}^2 \setminus \{0\}$.

Now we prove Theorem 1 by assuming Lemma 2. First we construct an axially symmetric solution for the case $\gamma = 0$, and then by rotating the solution, we obtain a solution for all γ. Let X be defined by

$$X := \{u \in C(\overline{\tilde{\Omega}_\varepsilon}); u(x_1, -x_2) = u(x_1, x_2) \text{ for } (x_1, x_2) \in \tilde{\Omega}_\varepsilon\}.$$

The reason why we consider axially symmetric function is to take away the rotational invariance of the equation (3.3). Lemma 2 and the Fredholm alternative show that for any $f \in X$, there exists a unique weak solution $\Psi \in H_0^1(\tilde{\Omega}_\varepsilon)$ of the equation $\mathcal{L}_{\varepsilon,k,\rho,0}(\Psi) = f$ such that Ψ has axially symmetry about x_1-axis. By the elliptic regularity theory, we have $\Psi \in X$. Thus we can define the operator $T : X \ni f \mapsto \Psi \in X$ and the estimate

$$\|Tf\|_{L^\infty(\tilde{\Omega}_\varepsilon)} \leq C\left(\log\frac{1}{\varepsilon}\right)\|\eta_k f\|_{L^\infty(\tilde{\Omega}_\varepsilon)}$$

holds for $f \in X$. From this inequality and (3.4), one can show that the mapping $X \ni v \mapsto -TF_{\varepsilon,k,\rho,0}(v) \in X$ is a contraction mapping in $\{u \in X; \|u\|_{L^\infty(\tilde{\Omega}_\varepsilon)} \leq C\varepsilon^{k-1}\log(1/\varepsilon)\}$ for some $C > 0$ depending only on δ and sufficiently small ε. Thus we obtain the desired solution.

What is left is to prove Lemma 2. One of the keys to proving the lemma is to determine the kernel of the limiting operator of $\mathcal{L}_{\varepsilon,k,\rho,\gamma}$ as $\varepsilon \to 0$. This is defined by

$$\mathcal{L}_{0,k,\rho,\gamma} := \Delta + 8k^2(1-\rho^2)e^{v_{k,\rho,\gamma}} = \Delta + \frac{8k^2(1-\rho^2)}{r^2\{r^k + r^{-k} - 2\rho\cos(k\theta+\gamma)\}^2},$$

which operates on functions defined on $\mathbb{R}^2 \setminus \{0\}$. It is easy to see that the functions

$$\Phi_{k,\rho,\gamma,1}(x) = \frac{r^k - r^{-k}}{r^k + r^{-k} - 2\rho\cos(k\theta+\gamma)}\left(= -\frac{1}{2k}(x\cdot\nabla v_{k,\rho,\gamma}(x) + 2)\right),$$
$$\Phi_{k,\rho,\gamma,2}(x) = \frac{2\cos(k\theta+\gamma) - \rho(r^k + r^{-k})}{r^k + r^{-k} - 2\rho\cos(k\theta+\gamma)}\left(= \frac{2}{1-\rho^2}\frac{\partial}{\partial\rho}\{v_{k,\rho,\gamma}(x) + \log(1-\rho^2)\}\right),$$
$$\Phi_{k,\rho,\gamma,3}(x) = \frac{\sin(k\theta+\gamma)}{r^k + r^{-k} - 2\rho\cos(k\theta+\gamma)}\left(= \frac{1}{4\rho}\frac{\partial}{\partial\gamma}v_{k,\rho,\gamma}(x)\right)$$

are bounded and satisfy $\mathcal{L}_{0,k,\rho,\gamma}\Phi_{k,\rho,\gamma,j} = 0$ for $j = 1, 2, 3$. Moreover, it can be shown that there is no linearly independent bounded function in the kernel of $\mathcal{L}_{0,k,\rho,\gamma}$. In fact, the following lemma holds.

Lemma 3 ([4], [6]). *Let $\Phi \in L^\infty(\mathbb{R}^2)$ satisfy $\mathcal{L}_{0,k,\rho,\gamma}\Phi = 0$. Then Φ is a linear combination of $\Phi_{k,\rho,\gamma,1}$, $\Phi_{k,\rho,\gamma,2}$ and $\Phi_{k,\rho,\gamma,3}$.*

In what follows, we briefly show Lemma 2. We prove by contradiction. Suppose that (3.5) does not hold. Then there exist sequences $\{\Psi_j\}_{j=1}^\infty$, $\{\varepsilon_j\}_{j=1}^\infty$, $\{k_j\}_{j=1}^\infty$, $\{\rho_j\}_{j=1}^\infty$ and

$\{\gamma_j\}_{j=1}^{\infty}$ such that

$$\|\Psi_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})} = 1, \quad \left(\log \frac{1}{\varepsilon_j}\right) \|\eta_k f_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})} \to 0,$$
$$\varepsilon_j \to 0, \quad k_j \to k_0 \in [2, \infty], \quad \rho_j \to \rho_0 \in (0,1), \quad \gamma_j \to \gamma_0 \in S^1$$

as $j \to \infty$, where $f_j := \mathcal{L}_{\varepsilon_j,k_j,\rho_j,\gamma_j}(\Psi_j)$. We only treat the case $k_0 < +\infty$ here. We can also derive a contradiction for the case $k_0 = +\infty$.

Suppose that $k_0 < +\infty$. Then the L^p estimate for the elliptic operator and the Sobolev embedding theorem show that a subsequence of $\{\Psi_j\}_{j=1}^{\infty}$ (we denote it by the same notation $\{\Psi_j\}_{j=1}^{\infty}$) converges to some function Ψ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$. Furthermore Ψ must satisfy $\|\Psi\|_{L^\infty(\mathbb{R}^2)} \le 1$, $\mathcal{L}_{0,k_0,\rho_0,\gamma_0}(\Psi) = 0$ and $\langle \Psi, \Phi_{k_0,\rho_0,\gamma_0,3}\rangle_{L^2(\mathbb{R}^2,|x|^{-2}dx)} = 0$. From Lemma 3, these implies that $\Psi = c_1\Phi_{k_0,\rho_0,\gamma_0,1} + c_2\Phi_{k_0,\rho_0,\gamma_0,2}$ for some $c_1, c_2 \in \mathbb{R}$.

Let φ_+ and φ_- be defined by

$$\varphi_\pm(x) := \alpha_\pm \log r + \beta_\pm - 2r^{\pm k},$$

where $\alpha_\pm$ and $\beta_\pm$ are determined by the relation

$$\varphi_+(R) = 1, \quad \varphi_+\left(\varepsilon_J^{\frac{1}{2}(1+\frac{1}{k})}\right) = 0,$$
$$\varphi_-(R^{-1}) = 1, \quad \varphi_-\left(\varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}\right) = 0$$

for $R < 1$. Then we can take R depending only on δ such that

$$\varphi_+ > 0, \quad \mathcal{L}_{\varepsilon_j,k_j,\rho_j,\gamma_j}\varphi_+ \le -k^2 r^{k-2} \quad \text{in} \quad \left\{\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})} < |x| < R\right\},$$
$$\varphi_- > 0, \quad \mathcal{L}_{\varepsilon_j,k_j,\rho_j,\gamma_j}\varphi_- \le -k^2 r^{-k-2} \quad \text{in} \quad \left\{R^{-1} < |x| < \varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}\right\}$$

for large j. In particular, this shows that the maximum principle holds for the operator $\mathcal{L}_{\varepsilon_j,k_j,\rho_j,\gamma_j}$ on $\{\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})} \le |x| \le R\}$ and $\{R^{-1} \le |x| \le \varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}\}$. Moreover, from the maximum principle, we have

$$|\Psi_j(x)| \le \max\left\{\sup_{|x|=R} |\Psi_j(x)|, 2\|\eta f_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})}\right\}\varphi_+(x)$$
$$\le C\max\left\{\sup_{|x|=R} |\Psi_j(x)|, 2\|\eta f_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})}\right\}$$

for $\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})} \le |x| \le R$ and

$$|\Psi_j(x)| \le \max\left\{\sup_{|x|=R^{-1}} |\Psi_j(x)|, 2\|\eta f_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})}\right\}\varphi_-(x)$$
$$\le C\max\left\{\sup_{|x|=R^{-1}} |\Psi_j(x)|, 2\|\eta f_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})}\right\}$$

for $R^{-1} \le |x| \le \varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}$. This implies that if $c_1 = c_2 = 0$, then $\|\Psi_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})} \to 0$. This contradicts the fact that $\|\Psi_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})} = 1$, and therefore it is enough to show that $c_1 = c_2 = 0$.

We prove by contradiction that $c_1 = c_2 = 0$. First we assume that $c_1 + c_2 > 0$ and $c_1 - c_2 > 0$ and derive a contradiction. Since

$$\Phi_{k,\rho,\gamma,1}(r,\theta) \to -1, \quad \Phi_{k,\rho,\gamma,2}(r,\theta) \to 1 \quad \text{uniformly for } \theta \in S^1 \text{ as } r \to 0,$$
$$\Phi_{k,\rho,\gamma,1}(r,\theta) \to 1, \quad \Phi_{k,\rho,\gamma,2}(r,\theta) \to 1 \quad \text{uniformly for } \theta \in S^1 \text{ as } r \to \infty,$$

we see that $m_\pm := \inf_{\theta\in S^1} \Psi_j(R^{\pm 1},\theta) \ge (c_1 \mp c_2)/2$ for small R and large j. We introduce the comparison functions ψ_+ and ψ_- defined by

$$\psi_\pm(x) := \tilde{\alpha}_\pm \log r + \tilde{\beta}_\pm + 2r^{\pm k}.$$

Here $\tilde{\alpha}_\pm$ and $\tilde{\beta}_\pm$ are determined by solving the equations

$$\psi_+(R) = 1, \quad \psi_+\left(\varepsilon_J^{\frac{1}{2}(1+\frac{1}{k})}\right) = 0,$$
$$\psi_-(R^{-1}) = 1, \quad \psi_-\left(\varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}\right) = 0.$$

Then it can be checked that

$$\tilde{\alpha}_+ \ge \frac{1}{2}\left(\log\frac{1}{\varepsilon_j}\right)^{-1}, \quad \tilde{\alpha}_- \le -\frac{1}{2}\left(\log\frac{1}{\varepsilon_j}\right)^{-1},$$
$$\mathcal{L}_{\varepsilon_j,k_j,\rho_j,\gamma_j}\psi_+ \ge k^2 r^{k-2} \quad \text{on} \quad \left\{\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})} \le |x| \le R\right\},$$
$$\mathcal{L}_{\varepsilon_j,k_j,\rho_j,\gamma_j}\psi_- \ge k^2 r^{-k-2} \quad \text{on} \quad \left\{R^{-1} \le |x| \le \varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}\right\}$$

provided that j is large. The maximum principle gives

$$\Psi_j(x) \ge \frac{1}{2}(c_1 - c_2)\psi_+(x)$$

for $\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})} \le |x| \le R$ and

$$\Psi_j(x) \ge \frac{1}{2}(c_1 + c_2)\psi_-(x)$$

for $R^{-1} \le |x| \le \varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}$. In particular, we have

$$r\frac{\partial \Psi_j}{\partial r}\bigg|_{r=\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})}} \ge r\frac{d\psi_+}{dr}\bigg|_{r=\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})}} \ge \tilde{\alpha}_+ \ge \frac{1}{2}(c_1 - c_2),$$
$$r\frac{\partial \Psi_j}{\partial r}\bigg|_{r=\varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}} \le r\frac{d\psi_-}{dr}\bigg|_{r=\varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}} \le \tilde{\alpha}_- \le -\frac{1}{2}(c_1 + c_2).$$

Multiplying $\mathcal{L}_{\varepsilon_j,k_j,\rho_j,\gamma_j}(\Psi_j) = f_j$ by Ψ and integrating over $\tilde{\Omega}_\varepsilon$ yield

$$\int_0^{2\pi} \left[r\Psi(r,\theta)\frac{\partial \Psi_j}{\partial r}(r,\theta) \right]_{r=\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})}}^{\varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}} d\theta = o\left(\left(\log \frac{1}{\varepsilon} \right)^{-1} \right).$$

Since the right hand side can be estimated as

$$\int_0^{2\pi} \left[r\Psi(r,\theta)\frac{\partial \Psi_j}{\partial r}(r,\theta) \right]_{r=\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})}}^{\varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}} d\theta \le -\frac{1}{8}\left\{ (c_1 - c_2)^2 + (c_1 + c_2)^2 \right\} \left(\log \frac{1}{\varepsilon_j} \right)^{-1},$$

we conclude that $(c_1 - c_2)^2 + (c_1 + c_2)^2 = 0$, which gives a contradiction.

Next we consider the case $c_1 - c_2 = 0$ and $c_1 + c_2 > 0$. By using the comparison function φ_+, we have

$$\left| r\frac{\partial \Psi_j}{\partial r}\bigg|_{r=\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})}} \right| \le \max\left\{ \sup_{|x|=R} |\Psi_j(x)|, 2\|\eta f_j\|_{L^\infty(\tilde{\Omega}_{\varepsilon_j})} \right\} \left| r\frac{d\varphi_+}{dr}\bigg|_{r=\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})}} \right|$$
$$= o\left(\left(\log \frac{1}{\varepsilon} \right)^{-1} \right).$$

Hence, in this case,

$$\int_0^{2\pi} \left[r\Psi(r,\theta)\frac{\partial \Psi_j}{\partial r}(r,\theta) \right]_{r=\varepsilon_j^{\frac{1}{2}(1+\frac{1}{k})}}^{\varepsilon_j^{-\frac{1}{2}(1-\frac{1}{k})}} d\theta \le -\frac{1}{8}\left\{ (c_1 + c_2)^2 + o(1) \right\} \left(\log \frac{1}{\varepsilon_j} \right)^{-1}.$$

as $j \to 0$. This implies that $c_1 + c_2 = 0$, and a contradiction is derived.

The other cases can be treated in a similar way. Thus we conclude that $c_1 = c_2 = 0$, and the Lemma 2 is proved.

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